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UNIQUENESS RESULTS FOR HOMEOMORPHISM GROUPS

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ABSTRACT. Let X be a separable metric manifold and let $\mathcal{H}(X)$ be the homeomorphism group of X. Then $\mathcal{H}(X)$ has a unique topology in which it is a complete separable metric group. Similar results are demonstrated for a much wider class of spaces, X, and for many subgroups of the homeomorphism group.

1. Introduction. Throughout the space X will be assumed to be Hausdorff with a countable basis for its topology and $\mathcal{H}(X)$ will denote the homeomorphism group of X. If G is an abstract subgroup of $\mathcal{H}(X)$, the symbol $G < \mathcal{H}(X)$ will mean that G is a topological group in some complete separable metric topology so that for each x in X the mapping $G \to X$ given by $g \to g(x)$ is continuous.

The major result of this paper is the following theorem.

THEOREM 1.1. *Assume X does not have exactly two isolated points and that $G < \mathcal{H}(X)$. Suppose further that for each open subset U of X which does not consist of a single point, there exists some element g_U in G so that g_U is not the identity on X, but g_U is the identity on $X - U$. Let H be a complete separable metric group and let $\psi\colon H \to G$ be an abstract group isomorphism. Then ψ is a topological isomorphism.*

Note that an instant corollary of this theorem is that every automorphism of G is continuous. Therefore, this theorem really says something about G, for it is false for G the additive group of the reals or the circle group. There are some special results in the literature which suggest that Theorem 1.1 might be true. For example, if X is the integers, then $\mathcal{H}(X)$ is S_∞, the group of all permutations of the integers. Theorem 1.1 in this special case is the main result of Kallman [**6**], which answered a question posed by Ulam, Schreier, and von Neumann. Whittaker [**12**] studied the following question: If X and Y are topological spaces and $\psi\colon \mathcal{H}(X) \to \mathcal{H}(Y)$ is an abstract group isomorphism, does there exist a homeomorphism $\omega\colon X \to Y$ which implements ψ? He showed that this question has an affirmative answer if X and Y are compact manifolds, with or without boundary. As Whittaker noted, such a result cannot hold in general without a compactness assumption on both spaces—just consider the natural restriction mapping of $\mathcal{H}([0,1]) \to \mathcal{H}((0,1))$. This is an abstract group isomorphism which certainly cannot be implemented by an ω. Note, however, that the conclusion of Theorem 1.1 holds in this case. Filipkiewicz [**4**] recently proved an analogue of Whittaker's result for $\mathrm{Diff}^p(X)$ and $\mathrm{Diff}^q(Y)$, if X and Y are separable metric C^p- and C^q-manifolds without boundary.

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Theorem 1.1 is proved in §2. The reason for the somewhat bizarre restriction on X is discussed in §3. A variety of corollaries of Theorem 1.1 is derived in §4. In particular, Theorem 1.1 holds for the homeomorphism group of any separable metric manifold, with or without boundary or corners; for the group $\mathrm{Diff}^p(X)$, for X a compact metric C^p-manifold; for the homeomorphism group of a connected countable locally finite simplicial complex; for the homeomorphism group of the Hilbert cube; and for the homeomorphism group of the Cantor set. Another easy corollary then is that none of the groups mentioned above has a σ-finite Borel measure which is quasi-invariant under left translations. Open questions and counterexamples are discussed in §5.

2. Proof of Theorem 1.1. The proof is a consequence of the theory of functions with the Baire property, which may be found in Kuratowski [**8**]. The fact that X may have isolated points is a bit of a complicating factor. Note that if X has only one isolated point, then every element of $\mathcal{H}(X)$ leaves it fixed. Nothing is lost by tossing out this isolated point and viewing $\mathcal{H}(X)$ as operating on the complement.

LEMMA 2.1. *Let U be a nonempty open subset of X which either contains no isolated point or contains more than two isolated points. Suppose g in G commutes with every g_W, where W is an open subset of U which contains more than one point. Then $g|\overline{U}$ is the identity.*

PROOF. It suffices to show that $g|U$ is the identity. First, if U has any isolated points, then g must leave them fixed. To see this, note that if W consists of two isolated points, then g_W must be the transposition of them. If W consists of the two isolated points a and b of U, then $gg_W = g_W g$ implies that g maps W into itself. For suppose that $g(a)$ is not a or b. Then $g(b) = gg_W(a) = g_W g(a) = g(a)$, which implies that $a = b$, contradiction. Hence, if W consists of three isolated points, then g maps W into itself and $g|W$ is a central element of S_3, the permutation group on three elements. Therefore, $g|W$ must be the identity, and g leaves every isolated point of U fixed. Hence, we may assume that U has no isolated point.

If $g|U$ is not the identity, there is a nonempty open subset W of U so that $g(W) \cap W = \varnothing$. Choose x in W so that $g_W(x) \neq x$. Then $gg_W(x) = g_W g(x) = g(x)$ since $g(x)$ is not in W. This implies that $g_W(x) = x$, contradiction. Hence, $g|U$ is the identity. This proves Lemma 2.1.

Let U and V be nonempty open subsets of X. Define $C(U, V) = [g \text{ in } G \mid g(\overline{U})$ is contained in $\overline{V}]$. $C(U, V)$ is closed in G since the mapping $g \to g(x)$ is continuous for every x in X. Suppose further that U contains either no isolated point or more than two isolated points, and that $X - \overline{V}$ contains either no isolated point or more than two isolated points. Then $C(U, V) = \bigcap_{U', V'} [g \text{ in } G \mid gg_{U'}g^{-1}$ commutes with $g_{V'}]$, where U' ranges over the nonempty open subsets of U with more than one point, and V' ranges over the nonempty open subsets of $X - \overline{V}$ with more than one point. To see this, note that $gg_{U'}g^{-1}$ and $g_{V'}$ commute for every U' and V' if and only if $gg_{U'}g^{-1}|X - \overline{V}$ is the identity for every U' by Lemma 2.1. But this is the case if and only if $g_{U'}g^{-1}|X - \overline{V} = g^{-1}|X - \overline{V}$ for every U', if and only if $g^{-1}(X - \overline{V}) \cap U = \varnothing$, since the set of points moved by some $g_{U'}$ is dense in U, if and only if $g^{-1}(X - \overline{V})$ is contained in $X - U$, if and only if $g(U)$ is contained in $\overline{V}$, if and only if $g(\overline{U})$ is contained in $\overline{V}$.

Let U_i $(i \geq 1)$ be a basis for the topology of X, which we can assume is closed under finite intersections and finite unions. Until further notice, assume that either X has no isolated point or has infinitely many isolated points. The sets $C(U_i, U_j)$ $(i, j \geq 1)$, where U_i either contains no isolated point or more than two, and where $X - \overline{U}_j$ contains either no isolated point or more than two, separate the points of G. For suppose that g_1 and g_2 are two distinct elements of G. The set $U = [x \text{ in } X | g_1(x) \neq g_2(x)]$ is nonempty and open in X. If U contains a nonisolated point, then for some i, j, and k, $g_1(U_i)$ is contained in U_j, $g_2(U_i)$ is contained in U_k, $U_j \cap U_k = \varnothing$, U_i either has no isolated point or infinitely many, and $X - \overline{U}_k$ either has no isolated point or infinitely many. Hence, g_2 is in $C(U_i, U_k)$ and g_1 is not in $C(U_i, U_k)$.

Next, suppose that $[x \text{ in } X \mid g_1(x) \neq g_2(x)]$ consists solely of isolated points. Suppose that $g_1(a) \neq g_2(a)$. Choose points b and c so that $\{b, c, g_1(b), g_1(c), g_2(b), g_2(c)\} \cap \{a, g_1(a), g_2(a)\} = \varnothing$. This certainly is possible since X has infinitely many isolated points. Suppose that $U_i = \{a, b, c\}$ and $U_j = \{g_1(a), g_1(b), g_1(c)\}$. Then g_1 is in $C(U_i, U_j)$ and g_2 is not in $C(U_i, U_j)$.

After these preparations, the proof of Theorem 1.1 in the case under consideration follows from the theory of functions with the Baire property (Kuratowski [**8**]). Suppose that H is a complete separable metric group and that $\psi\colon H \to G$ is an abstract group isomorphism. The sets $C(U_i, U_j)$, where U_i has either no isolated point or infinitely many, and where $X - \overline{U}_j$ has either no isolated point or infinitely many, are closed subsets of G, separate the points of G, and therefore generate the Borel structure of G (Mackey [**9**]). Furthermore,

$$C(U_i, U_j) = \bigcap_{U', V'} [g \text{ in } G \mid g g_{U'} g^{-1} \text{ commutes with } g_{V'}],$$

where U' ranges over the nonempty open subsets of U_i with more than one point, and V' ranges over the nonempty open subsets of U_j with more than one point. Hence,

$$\psi^{-1}(C(U_i, U_j)) = \bigcap_{U', V'} [h \text{ in } H \mid h\psi^{-1}(g_{U'})h^{-1} \text{ commutes with } \psi^{-1}(g_{V'})],$$

where U' and V' are as above. But each set of the form $[h \text{ in } H | h\psi^{-1}(g_{U'})h^{-1}$ commutes with $\psi^{-1}(g_{V'})]$ is a closed subset of H. Hence, $\psi^{-1}(C(U_i, U_j))$ is a closed subset of H, and so if B is a Borel subset of G, then $\psi^{-1}(B)$ is a Borel subset of H. Therefore ψ is a Borel mapping. Next, apply a standard argument to show that $\psi\colon H \to G$ is a topological isomorphism, as follows. Results from Kuratowski [**8**] imply that there is a residual set H' in H such that $\psi | H'$ is continuous. It follows that ψ is actually continuous on all of H. To see this, let h_n $(n \geq 1)$ and h be elements of H so that $h_n \to h$. The union of $h^{-1}(H - H')$ and of the $h_n^{-1}(H - H')$ $(n \geq 1)$ is a set of the first category. Hence, there exists an element h' in the complement. Then hh' is in H' and $h_n h'$ is in H' $(n \geq 1)$. But $h_n h' \to hh'$ and so $\psi(h_n h') \to \psi(hh')$. Hence, $\psi(h_n) \to \psi(h)$. Hence, ψ is continuous. Souslin's theorem implies that $\psi^{-1}\colon G \to H$ is a Borel mapping, and so, repeating the above argument, ψ^{-1} is continuous. Hence, ψ is a topological isomorphism, and Theorem 1.1 is proved in case X has either no or infinitely many isolated points.

Next, suppose that X has a finite number, but more than two, isolated points. X is the union of F, the isolated points, and its complement X'. X' is open in

X. G maps F into itself, and G contains S, the symmetric group on F. Let G' be those elements of G which leave every element of F pointwise fixed. G' is a closed subgroup of G since the mapping, $G \to X$, $g \to g(x)$, is continuous for every x in X. Since F has three or more points, G' equals the centralizer of S in G. Note that $G = S \cdot G'$ and $S \cap G'$ is the identity. Hence, $G = S \times G'$ as an abstract group. The techniques used in the previous paragraph, for instance, show that $G = S \times G'$ as a topological group. $H' = \psi^{-1}(G')$ is the centralizer of $\psi^{-1}(S)$ in H. Hence, H' is closed in H and is a complete separable metric group. $H = \psi^{-1}(S) \cdot H'$, $\psi^{-1}(S) \cap H'$ is the identity, and so, reasoning as before, $H = \psi^{-1}(S) \times H'$ as a topological group. The triple X', G', and H' satisfies the hypotheses of Theorem 1.1 and X' has no isolated point. Therefore, $\psi\colon H' \to G'$ is a topological isomorphism by what has already been proved. Hence, $\psi\colon H = \psi^{-1}(S) \times H' \to S \times G' = G$ is a topological isomorphism. This proves Theorem 1.1.

3. The omitted case. Note that if X has only one isolated point, then every element of $\mathcal{H}(X)$ leaves it fixed. Nothing is lost by tossing out this isolated point and viewing $\mathcal{H}(X)$ as operating on the complement.

The case in which X has exactly two isolated points is a bit more complicated. Let the two isolated points be a and b. Then $\mathcal{H}(X)$ contains the transposition (ab) and S, the subgroup consisting of (ab) and the identity. The difficulty in proving Theorem 1.1 in this case lies in the fact that the centralizer of $\psi^{-1}(S)$ is not $\psi^{-1}(G')$, but in fact is all of H. It is crucial that there be some sort of algebraic control over the nature of $\psi^{-1}(G')$ in order to prove Theorem 1.1. This seemingly can be done only by additional assumptions.

PROPOSITION 3.1. *Suppose that X has exactly two isolated points. If G' is algebraically generated by its squares, then the conclusion of Theorem* 1.1 *holds. If G' is algebraically generated by its commutators, then the conclusion of Theorem* 1.1 *holds.*

PROOF. If G' is algebraically generated by its squares, then the same is true for $H' = \psi^{-1}(G')$. Since $H = \psi^{-1}(S) \cdot H'$ algebraically and $\psi^{-1}(S)$ is the center of H, the subgroup generated by the squares of elements of H coincides with H'. H' is an analytic subgroup of H which is of index two in H. Hence, H' is of second category in H. Banach [**2**, Théorème 1, p. 21] now implies that H' is both open and closed in H. The method of proof of Theorem 1.1 suffices to complete the proof in this case. A similar line of reasoning works if G' is algebraically generated by its commutators. This proves Proposition 3.1.

R. D. Anderson [**1**] has shown that many interesting homeomorphism groups are algebraically generated by commutators. See also his comments on Problem #29 of the Scottish Book [**10**].

4. Corollaries. For each integer $n \geq 1$, let X_n be a space which satisfies the hypotheses of Theorem 1.1, and let G_n either be the identity or a subgroup of $\mathcal{H}(X_n)$ which satisfies the hypotheses of Theorem 1.1. Let $G = \prod_{n\geq 1} G_n$. G in a natural manner is a complete separable metric group.

COROLLARY 4.1. *Let H be a complete separable metric group, and let $\psi\colon H \to G$ be an abstract group isomorphism. Then ψ is a topological isomorphism.*

PROOF. Note that each G_n is centerless by Lemma 2.1. For each integer $n \geq 1$, let $G'_n = \prod_{m \geq 1; m \neq n} G_m$, $H_n = \psi^{-1}(G_n)$ and $H'_n = \psi^{-1}(G'_n)$. G'_n is the centralizer of G_n in G, and G_n is the centralizer of G'_n in G. Hence, H'_n is the centralizer of H_n in H, and H_n is the centralizer of H'_n in H. Therefore, H_n and H'_n are closed subgroups of H. Since $H = H_n \cdot H'_n$ and $H_n \cap H'_n$ is the identity, $H = H_n \times H'_n$ as an abstract group, and therefore, as before, $H = H_n \times H'_n$ as a topological group. $\psi\colon H_n \to G_n$ is a topological isomorphism by Theorem 1.1. Let U be open in G_n. Then $U \times G'_n$ is a subbasic open set in G, and $\psi^{-1}(U \times G'_n) = \psi^{-1}(U) \times H'_n$ is open in H. Hence, ψ is continuous. As in the proof of Theorem 1.1, this implies that ψ is a topological isomorphism. This proves Corollary 4.1.

The following two lemmas are needed for applications.

LEMMA 4.2. *Let X be locally compact. Then $\mathcal{H}(X)$ can be given a complete separable metric topology which is at least as strong as the compact open topology. If X does not have exactly two isolated points, and if for every open subset U of X which consists of more than one point, there exists some element g_U of $\mathcal{H}(X)$ which is not the identity but is the identity outiside of U, then this given topology on $\mathcal{H}(X)$ is the only one which makes $\mathcal{H}(X)$ into a complete separable metric group. If X in addition is locally connected, then this given topology coincides with the compact open topology.*

This lemma should settle once and for all what is the "correct" topology to put on $\mathcal{H}(X)$ for most spaces X.

The first assertion of this lemma certainly must be known. The following is a condensed sketch of its proof, which is included because of its central importance for applications. First, suppose X is a compact metric space. X can be embedded into C, the Hilbert cube. Let ρ be a metric on X compatible with this embedding. Let X^X be the space of continuous functions from X to X with the metric $d(f,g) = \sup_x \rho(f(x), g(x))$. X^X can be embedded into C^X. The space $\mathcal{C}(X)$, the continuous real-valued functions on X, is separable in the sup norm metric since X is compact metric. Hence, C^X is separable, and so X^X and $X^X \times X^X$ are complete separable metric spaces. The mappings $(f,g) \to fg$, $(f,g) \to (g,f)$, and $(f_1,g_1) \times (f_2,g_2) \to (f_1 f_2, g_2 g_1)$ are continuous. Hence, the set of pairs (f,g) so that fg and gf are the identity is closed in $X^X \times X^X$. This set consists of all pairs (f, f^{-1}), where f is a homeomorphism of X onto X. The above comments show that this set is a complete separable metric group which is algebraically isomorphic to $\mathcal{H}(X)$. Give $\mathcal{H}(X)$ this complete separable metric group topology. Next, if X is locally compact with a countable basis for its topology, its one-point compactification X^* is a compact metric space, and $\mathcal{H}(X)$ may be identified with the closed subgroup of $\mathcal{H}(X^*)$ which leaves the point at infinity $*$ fixed. Hence, in this case also, $\mathcal{H}(X)$ may be made into a complete separable metric group. It is easy to check that this complete separable metric topology is at least as strong as the compact open topology.

The second assertion of the lemma follows from Theorem 1.1.

The third assertion follows from the second, for under the stated hypotheses $\mathcal{H}(X)$ is a complete separable metric group in the compact open topology (Gleason and Palais [**5**, Proposition 5.5]). This proves Lemma 4.2.

LEMMA 4.3. *Let X be a compact metric C^p-manifold and $G = \mathrm{Diff}^p(X)$ $(1 \leq p \leq \infty)$. Then G can be made into a complete separable metric group so that the natural injection of G into $\mathcal{H}(X)$ is continuous.*

PROOF. Let $\mathcal{C}^p(X,X)$ be the set of p times differentiable mappings of X into itself. $\mathcal{C}^p(X,X)$ is a complete separable metric space in a topology which is stronger than the topology on X^X. This is discussed on pages 19–23 of Palis and de Melo [**11**] for finite p. For $p = \infty$, the same result holds, for $\mathcal{C}^\infty(X,X)$ may be identified with the diagonal in the infinite product $\prod_{p\geq 1} \mathcal{C}^p(X,X)$. One checks easily that the mapping $(f,g) \to fg$, $\mathcal{C}^p(X,X) \times \mathcal{C}^p(X,X) \to \mathcal{C}^p(X,X)$, is continuous in these topologies. One now completes the proof of this lemma just as one did the proof of Lemma 4.2.

Theorem 1.1 and Corollary 4.1 apply to a great many spaces and groups of homeomorphisms. The following corollary lists some of the more obvious examples.

COROLLARY 4.4. *Theorem* 1.1 *holds for the following spaces and homeomorphisms*:

(a) *X is a connected countable locally finite simplicial complex and $G = \mathcal{H}(X)$*;

(b) *X is a separable metric locally Euclidean space, with or without boundary or corners, and $G = \mathcal{H}(X)$*;

(c) *X is a separable metric locally Euclidean space, with boundary or corners, and G is the closed subgroup of $\mathcal{H}(X)$ which leaves every point of the boundary fixed*;

(d) *X is the Hilbert cube and $G = \mathcal{H}(X)$*;

(e) *X is the Cantor set and $G = \mathcal{H}(X)$*;

(f) *X is a compact metric C^p-manifold and $G = \mathrm{Diff}^p(X)$ $(1 \leq p \leq \infty)$.*

PROOF. In every instance the space under consideration is locally compact with a countable basis and has no isolated point. Lemma 4.2 and Lemma 4.3 imply that in every instance the group under consideration is complete separable metric and the mapping $g \to g(x)$, $G \to X$, is continuous for every x in X. It is an elementary exercise in every instance that if U is a nonempty open subset of X, there is some element g_U in G so that g_U is not the identity but is the identity outside of U, for notice that one only has to check this condition for a basis of open sets. This proves Corollary 4.4.

One can combine Corollary 4.4 with Corollary 4.1 to prove a result about the countable products of the indicated groups.

The following folk theorem is needed for a discussion of quasi-invariant measures on the groups listed in Corollary 4.4. A proof is supplied because of the importance of its applications.

LEMMA 4.5. *None of the groups listed in Corollary* 4.4 *is locally compact.*

PROOF. For the examples (a), (b), and (c), G contains as a closed subgroup the group G' of homeomorphisms of some closed ball $B(0;1)$ in some R^n which leaves the boundary of B pointwise fixed. It suffices to show that G' is not locally compact. If G' were locally compact, then

$$\left[g \text{ in } G' \mid \sup_{x \text{ in } B} |g(x) - x| + \sup_{x \text{ in } B} |g^{-1}(x) - x| < \varepsilon\right]$$

would have compact closure in G' for some small positive ε. But then the closed subgroup G'' of G', consisting of all homeomorphisms of the closed ball $B(0;\varepsilon/6)$

which leave the boundary pointwise fixed, would be compact. But clearly $G''(0)$ is the open ball about 0 of radius $\varepsilon/6$, contradiction. Hence, none of the groups listed in (a), (b), or (c) is locally compact. In example (d) X is the Hilbert cube and X is homeomorphic to $X \times I$, where I is the closed unit interval. Hence, $\mathcal{H}(X)$ contains the homeomorphism group of I as a closed subgroup, which has just been shown to be not locally compact. In example (e), X is the Cantor set, which may be viewed as a countable product of spaces, each consisting of two elements, say 0 and 1. S_∞, the group of permutations of the positive integers, may be embedded in $\mathcal{H}(X)$ as a closed subgroup by allowing S_∞ to permute the coordinates of this infinite product. S_∞ is not locally compact, for a basic neighborhood of the identity consists of all permutations of the positive integers which leave a finite block of positive integers pointwise fixed. Such a basic neighborhood is an open and closed subgroup of S_∞ which cannot be compact, for it has unbounded orbits. Hence, $\mathcal{H}(X)$ cannot be locally compact in this case also.

Finally, suppose $G = \mathrm{Diff}^p(X)$, where X is a compact C^p-manifold. If G is locally compact, then the set of C^p-diffeomorphisms of the closed unit ball, which are invariant under rotations and are the identity outside the shell whose inner radius is $\frac{1}{3}$ and whose outer radius is $\frac{2}{3}$, will be locally compact. But this latter group is topologically isomorphic to the group of C^p-diffeomorphisms of the line which are the identity outside of $[0,1]$. Let G' be this last group, and let Z be the set of C^p-functions on the line which are supported on $[0,1]$. Z is a Fréchet space in a well-known natural topology. Since Z is infinite dimensional, Z cannot be locally compact. Let $Z_{1/2} = [\lambda \text{ in } Z \mid \|\lambda'\|_\infty \leq \frac{1}{2}]$. $Z_{1/2}$ is a neighborhood of 0 in Z. The mapping $\lambda(x) \to x + \lambda(x)$, $Z_{1/2} \to G'$, is a topological isomorphism of $Z_{1/2}$ onto a closed neighborhood of the identity in G'. Hence, G' cannot be locally compact. This proves Lemma 4.5.

See Mackey [**9**] for the background on standard Borel groups used in the following corollary.

COROLLARY 4.6. *Let G be any one of the groups listed in Corollary* 4.4. *There is no analytic Borel structure $\mathcal{B}$ on G, with respect to which G is an analytic Borel group, and which admits a σ-finite Borel measure μ on $\mathcal{B}$, all of whose left translates under G have the same null sets. In particular, G cannot be given the structure of a locally compact group with a countable basis for its topology.*

Note that there is no a priori reason to believe that $\mathcal{B}$ has any connection with any topological group structure on G.

Suppose that there is some analytic Borel structure $\mathcal{B}$ on G, with respect to which G is an analytic Borel group, and which admits a σ-finite Borel measure μ on $\mathcal{B}$, all of whose left translates under G have the same null sets. Theorem 7.1 of Mackey [**9**] implies that G can be made into a locally compact group with a countable basis for its topology, and so that $\mathcal{B}$ is the set of Borel sets with respect to this topology. But Corollary 4.4 implies that G has a unique topology in which it is a complete separable metric group. Lemma 4.5 implies that this unique topology is not locally compact, contradiction. This proves Corollary 4.6.

5. Questions and comments. The results of §4 carry over to many other groups—certainly to the case in which X is a separable metric C^p-manifold, with or without boundary or corners, and in which G is the group of C^p-diffeomorphisms of X.

Does Theorem 1.1 still hold in case X has exactly two isolated points? For what locally compact X's with a countable basis does Theorem 1.1 hold for $\mathcal{H}(X)$? The method for attacking such a question must be different from the methods employed to prove Theorem 1.1, for g_U's need not exist in general. Is there an analogue of Theorem 1.1 if X is a real analytic manifold (perhaps even compact) and if G is the group of real analytic diffeomorphisms of X? Note that there are no g_U's in this case, for a real analytic diffeomorphism on a connected analytic manifold must certainly be determined by its action on any nonempty open set. There is no analogue of Theorem 1.1 for X a complex analytic manifold and for G the group of complex analytic diffeomorphisms of X. For example, if X is the Riemann sphere and G is the group of complex analytic diffeomorphisms of X, then G is isomorphic to $\mathrm{SL}_2(C)/(\pm 1)$, which is known to have discontinuous automorphisms since the complex numbers have continuumly many discontinuous automorphisms.

It is known that if X is a complete separable metric space and if G is a subgroup of $\mathcal{H}(X)$ which may be given a complete separable metric group topology such that $g \to g(x)$, $G \to X$, is continuous for all x, then the mapping $(g, x) \to g(x)$, $G \times X \to X$, is continuous (Theorem 1 of Chernoff and Marsden [**3**]). This was not needed for the proof of Theorem 1.1. However, this result does imply that the compact-open topology on G is weaker than its given topology (Kelley [**7**, Theorem 5, p. 223]) if X is a complete separable metric space.

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